Electrical and Electronics
Engineering
2024-2025
Master Semester 2

Course
Smart grids technologies
Admittance matrix calculus

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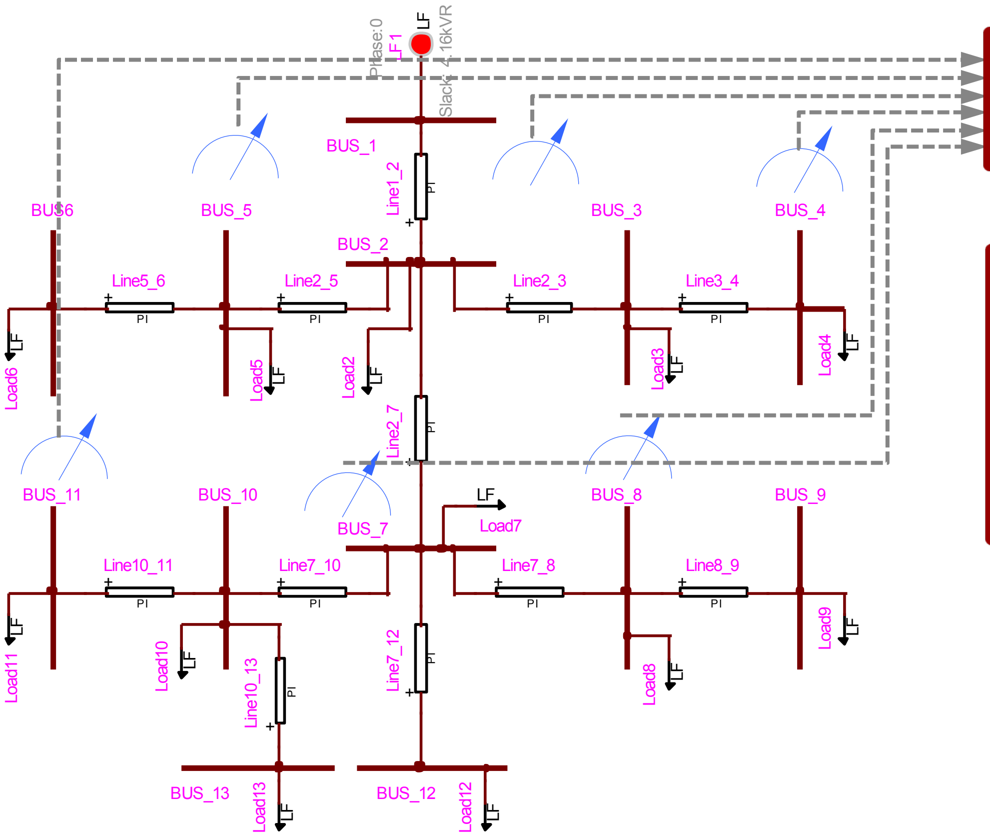
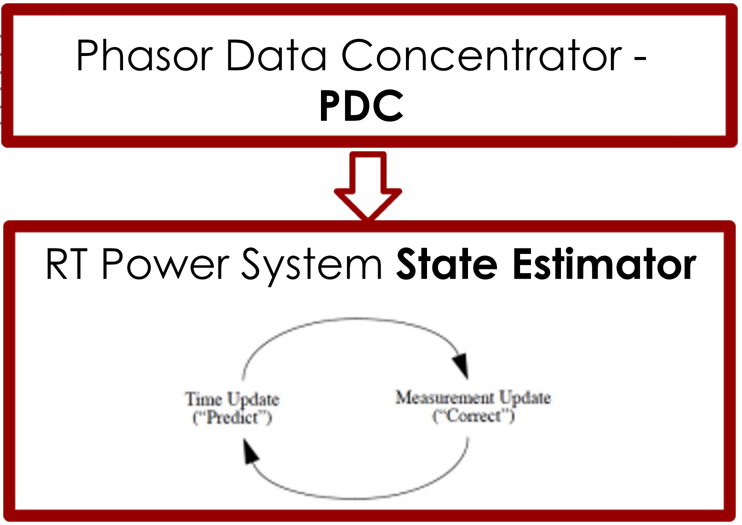
Preliminary remark on the use of PMUs

Why the knowledge of synchrophasors (estimated by PMUs) is of importance for power systems ?




Phasor Data Concentrator - **PDC**

RT Power System **State Estimator**

= Phasor Measurement Unit

Network

in normal operation

Voltage sensitivity

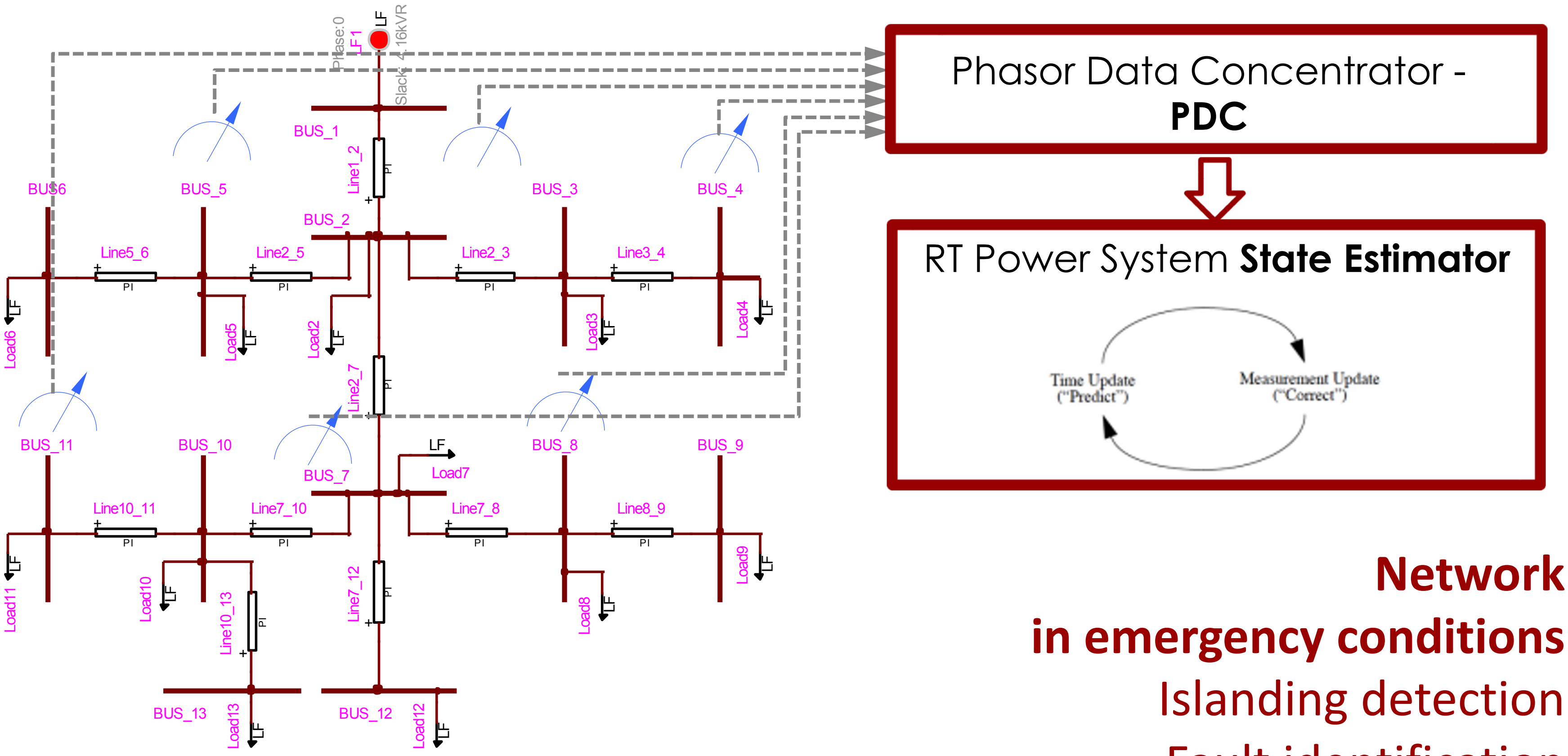
Power flows sensitivity

Voltage/Power optimal control

Real time congestion management

Preliminary remark on the use of PMUs

Why the knowledge of synchrophasors (estimated by PMUs) is of importance for power systems ?



= Phasor Measurement Unit

Network

in emergency conditions

Islanding detection

Fault identification

Fault location

Generation schedule

The knowledge of the network state

Observation:

If the network structure is known together with the admittances that compose it, **by knowing the voltages in module and phase in all the network nodes, it is possible to calculate all the other electrical quantities of interest, namely the power flows and currents injected or extracted from the nodes as well as the network losses and power/current flows along the lines**.

For this reason, the 'problem of **state estimation'** is practically **equivalent to the determination of the phasors representing the voltages in all the network nodes (state variables).**

Tools

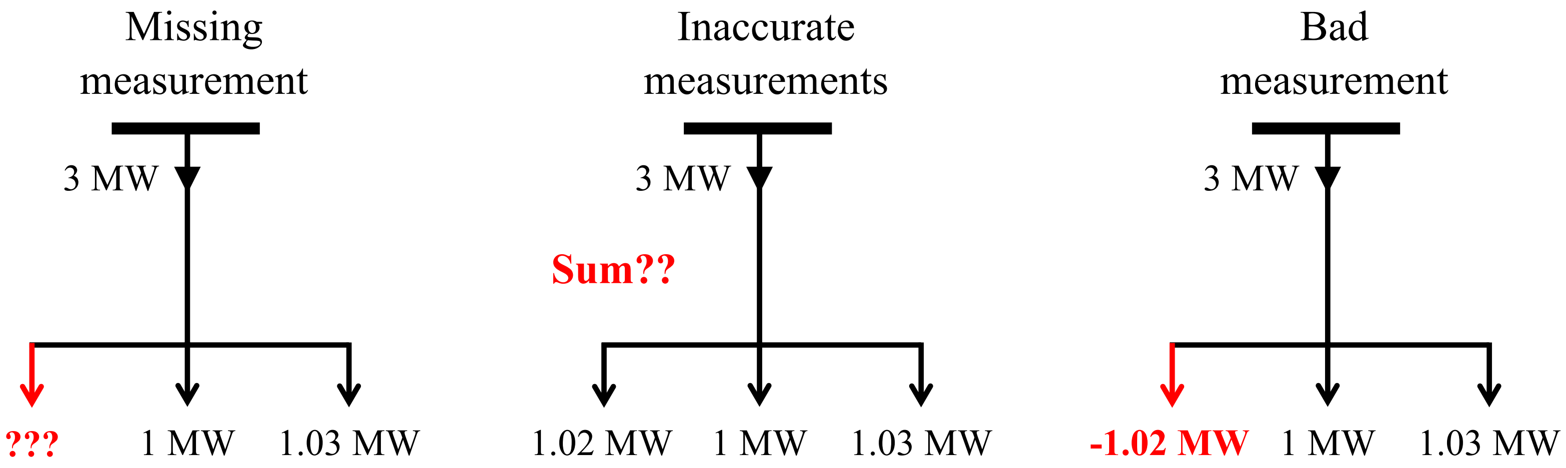
Which tools do we need to perform a power system state estimation ?

1. Electrical network admittance matrix calculus
2. Power flow (or load flow) computation engine
3. State estimation calculus

Tools: State Estimation vs Load Flow

Until the late seventies, conventional load flow calculations provided the system state by directly using the raw measurements of voltage magnitudes and injected powers. However, in Load Flow:

- the inputs are restricted to injected powers P, Q (at load buses) and P, V values (at voltage-controlled buses).
- In case that even **one of these inputs becomes unavailable**, it is **impossible to get a solution of the load flow problem.** Instead, the SE considers **redundant measurements.**
- Additionally, **gross errors** in one or more of the inputs makes the computation of the load flow problem **wrong**.



Tools: State Estimation vs. Load Flow

Therefore, load flow theory has been combined with statistical estimation forming the so-called **State Estimation**:

- ✓ Takes into account that the measurements (inputs) are **noisy**;
- ✓ uses **all types of measurements** (e.g., voltage and current magnitudes, nodal injected and line flow powers, synchrophasors) and evaluates their consistency using the network model;
- ✓ employs novel tools, such as **bad data detection and identification**, in order to find and replace erroneous or missing measurements;
- ✓ uses **redundant** information to improve its accuracy performance;
- ✓ solves an optimization problem in order to determine the **optimal estimate** of the system state (accurate, reliable).

The output of load flow and state estimation is composed of the same kind of quantities, typically the voltage magnitude and phase at all network buses.

From the Physical Network to the Admittance Matrix - Introduction

A power system is essentially composed of:

- ***buses*** (or nodes), that can be distinguished in ***generator buses*** (corresponding to the generator terminals), ***reactive compensation buses*** (corresponding to the terminals of the synchronous compensators and the static compensators), ***interconnection buses*** (where more lines converge in order to form the «meshed» configuration) and ***load buses*** (which feed the equivalent loads seen from the High Voltage network);
- ***capacitor banks*** in shunt and in series connection;
- ***transformers***;
- ***reactances***, in shunt and in series connection;
- ***lines*** overhead and cables that link the various buses;
- …

From the Physical Network to the Admittance Matrix - Introduction

Hypothesis:

- **network in permanent state of equilibrium,**
- **network topology and parameters are constant,**
- **constant load demands,**
- **electrical components are linear,**
- **the Network is symmetrical and balanced**

In view of the above hypothesis the phase-to-ground voltages and currents can be derived at every point of the network using the direct sequence. Their frequency correspond to the electrical speed of the synchronous machines and the active and reactive powers appear constant at every given point of the network.

Therefore, the three-phase network can be studied using an equivalent one-phase network.

From the Physical Network to the Admittance Matrix - Introduction

We will deal with this study using the **relative values** → the represented voltages in such an one-phase equivalent circuit are either the phase-to-ground or the phase-to-phase ones, therefore it is useful to apply the second ones given that the power flows in the circuit are the three-phase powers in per unit.

See lecture 2.2 and associated exercise.

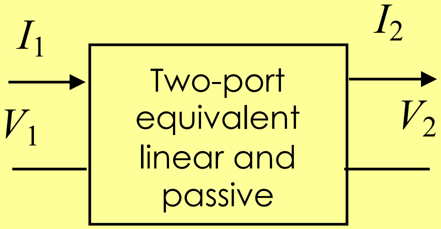
From the Physical Network to the Admittance Matrix – The Nodal Analysis

We consider a network **linear**, **passive** and **reciprocal**.

A network like this is characterized by $\boldsymbol{s}$**+1 buses**, with $\boldsymbol{g}$ **generator buses** and $\boldsymbol{u}$ **load buses**, where the bus $s+1$ is the 'neutral' (return conductor in Fig. 1).

The network is characterized by $\boldsymbol{m}$ **branches** each one having a unique series admittance. In general, the value of such an admittance is considered, for simplicity, independent from the assumed voltage and current values (namely we neglect the inductive couplings between neighbouring lines, the parameter variations - for example resistances – with the temperature and, therefore, with the ambient temperature and the current, etc.).

On the Two-Ports Network Equivalents



A two-port network equivalent is **passive** if, in 'open circuit', there are no voltages in the two couples of terminal ports.

$$V_1 = Z_{11}I_1 - Z_{12}I_2$$
$$V_2 = Z_{21}I_1 - Z_{22}I_2$$

$$\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

Impedance matrix in open circuit.

$$I_1 = Y_{11}V_1 - Y_{12}V_2$$
$$-I_2 = -Y_{21}V_1 + Y_{22}V_2$$

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & -Y_{12} \\ -Y_{21} & Y_{22} \end{bmatrix}$$

Admittance matrix in short circuit.

On the Two-Ports Network Equivalents

$$V_1 = AV_2 + BI_2$$
$$I_1 = CV_2 + DI_2$$

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$A = \frac{Z_{11}}{Z_{21}} = \frac{Y_{22}}{Y_{21}}$$

$$B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}} = \frac{1}{Y_{21}}$$

$$C = \frac{1}{Z_{21}} = \frac{Y_{11}Y_{22} - Y_{12}Y_{21}}{Y_{21}}$$

$$D = \frac{Z_{22}}{Z_{21}} = \frac{Y_{11}}{Y_{21}}$$

$$Z_{11} = \frac{A}{C}$$

$$Z_{12} = \frac{AD - BC}{C}$$

$$Z_{21} = \frac{1}{C}$$

$$Z_{22} = \frac{D}{C}$$

$$Z = -\frac{AD - BC - 1}{C}$$

$$Y_{11} = \frac{D}{B}$$

$$Y_{12} = \frac{AD - BC}{B}$$

$$Y_{21} = \frac{1}{B}$$

$$Y_{22} = \frac{A}{B}$$

$$Y = \frac{AD - BC - 1}{B}$$

On the Two-Ports Network Equivalents

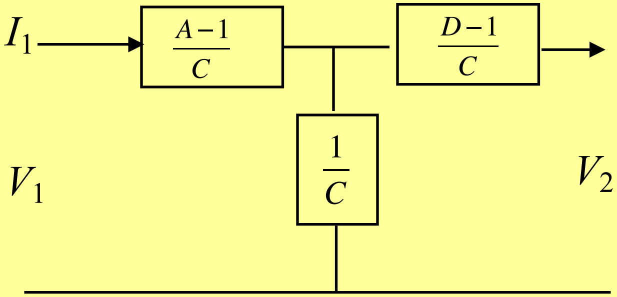
Two-port equivalent reciprocals:

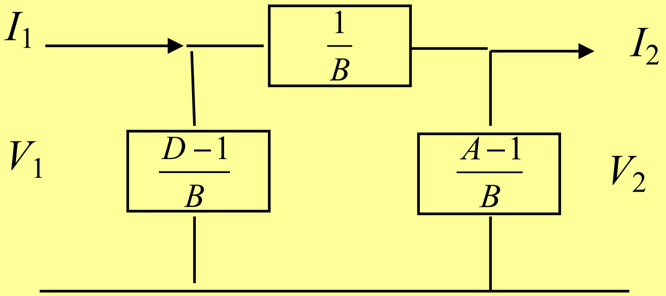
$$\left.\frac{I_1}{V_2}\right|_{V_1=0} = \left.\frac{-I_2}{V_1}\right|_{V_2=0}$$

$$AD - BC = 1$$

$$Z_{12} = Z_{21}$$

$$Y_{12} = Y_{21}$$





Symmetrical two-port network equivalents:

$$A = D$$

$$Z_{11} = Z_{22}$$

$$Y_{11} = Y_{22}$$

From the Physical Network to the Admittance Matrix – The Nodal Analysis

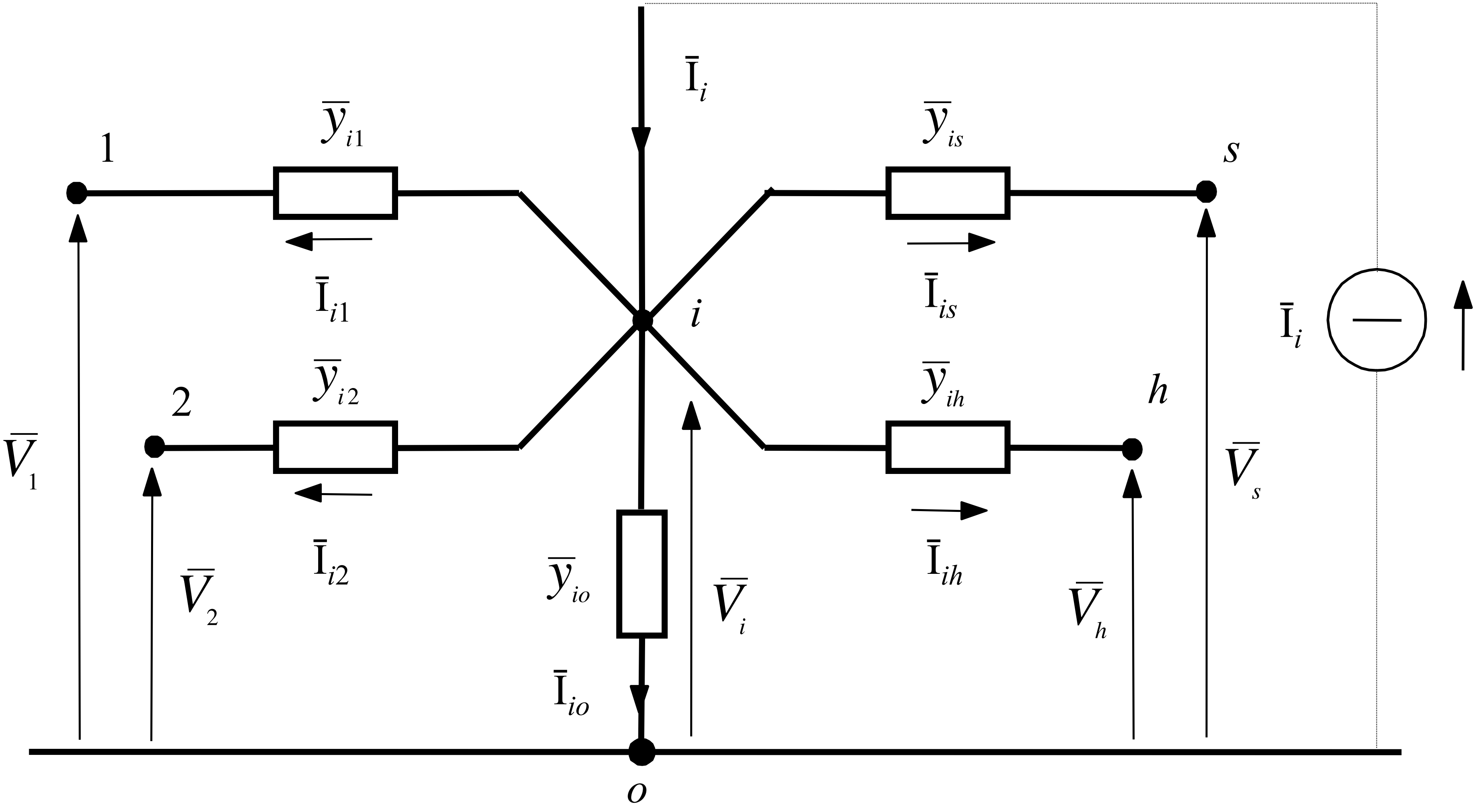


Fig. 1. Representation of the generic node *i* and its connections.

From the Physical Network to the Admittance Matrix – The Nodal Analysis

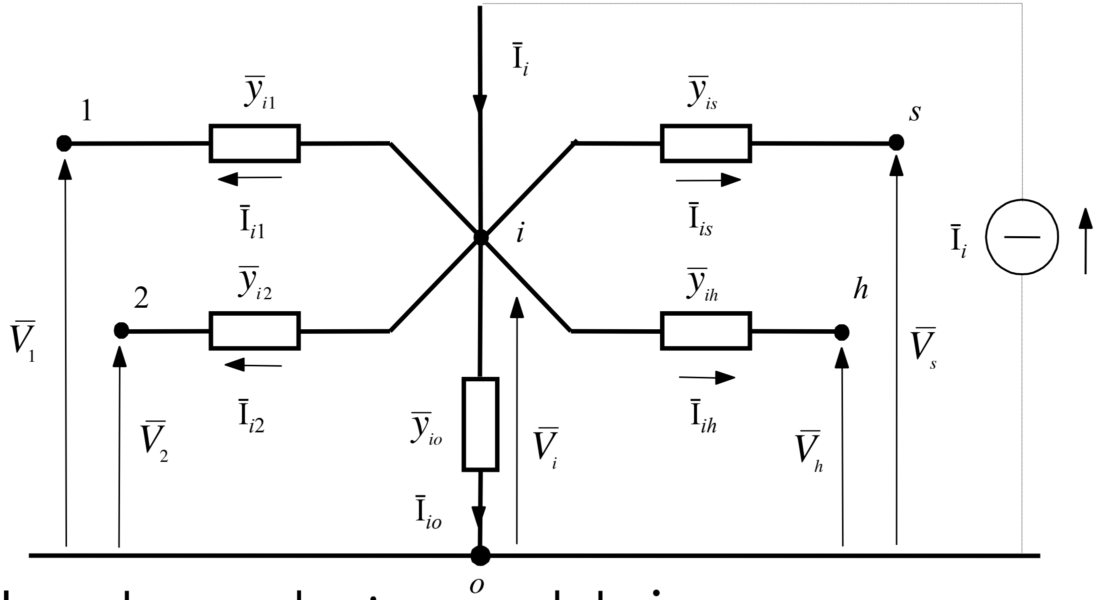
The problem that we would solve now (nodal analysis) is to find which are the relations between the node voltages and injected currents, considering the first as independent variables and the second as dependent variables.

For a generic network node i-th, the node voltage is indicated with $\bar{V}_i$ and with $\bar{I}_i$ the node-injected current (this last is the current delivered by a generator or absorbed by a load connected to the node). By convention, **a current that is injected by a generator** (into the network) **is considered with positive sign and a current absorbed by a load is considered with negative sign**. If a node has only the interconnection function (i.e. it does not have generators or loads connected to it), the corresponding injected current is, obviously, null. If more generators and loads are connected to a node, the node current is the algebraic sum of the corresponding complex currents. We indicate with $\bar{I}_{ih}$ the current of the branch that connects the nodes i and h. The complex admittance between node nodes i and h is indicated with $\bar{y}_{ih}$, whereas, with $\bar{y}_{io}$ we indicate the sum of the admittances existing between the node i and the neutral (ground).

From the Physical Network to the Admittance Matrix – The Nodal Analysis

With reference to the network represented in Fig. 1, the currents exiting from the node are:

$$\begin{cases} \bar{I}_{io} = \bar{y}_{io}\bar{V}_i \\ \bar{I}_{i1} = \bar{y}_{i1}(\bar{V}_i - \bar{V}_1) \\ \dots \\ \bar{I}_{is} = \bar{y}_{is}(\bar{V}_i - \bar{V}_s) \end{cases}$$



Applying the first Kirchhoff law to node i, we obtain:

$$\bar{I}_i = \bar{y}_{io}\bar{V}_i + \bar{y}_{i1}(\bar{V}_i - \bar{V}_1) + \ldots + \bar{y}_{is}(\bar{V}_i - \bar{V}_s) =$$

$$= (\bar{y}_{io} + \bar{y}_{i1} + \ldots + \bar{y}_{is})\bar{V}_i - \bar{y}_{i1}\bar{V}_1 - \ldots - \bar{y}_{is}\bar{V}_s =$$

$$= (\bar{y}_{io} + \bar{y}_{i1} + \ldots + \bar{y}_{is})\bar{V}_i - \sum_{\substack{\ell=1 \\ \ell \neq i}}^{s} \bar{y}_{i\ell}\bar{V}_\ell$$

From the Physical Network to the Admittance Matrix – The Nodal Analysis

By setting

$$\overline{Y}_{ii} = \overline{y}_{io} + \overline{y}_{i1} + \ldots + \overline{y}_{is} = \sum_{\ell=0}^{s} \overline{y}_{i\ell}$$

$$\overline{Y}_{i1} = -\overline{y}_{i1}$$

$$\ldots$$

$$\overline{Y}_{is} = -\overline{y}_{is}$$

we obtain

$$\overline{I}_{i} = \overline{Y}_{i1}\overline{V}_{1} + \overline{Y}_{i2}\overline{V}_{2} + \ldots + \overline{Y}_{ii}\overline{V}_{i} + \ldots + \overline{Y}_{is}\overline{V}_{s} = \sum_{\ell=1}^{s} \overline{Y}_{i\ell}\overline{V}_{\ell}$$

From the Physical Network to the Admittance Matrix – The Nodal Analysis

A similar equation can be written for all other nodes →

for the whole network
we can derive
the following system:

$$\begin{cases} \bar{I}_1 = \bar{Y}_{11}\bar{V}_1 + \ldots + \bar{Y}_{1h}\bar{V}_h + \ldots + \bar{Y}_{1s}\bar{V}_s \\ \ldots \\ \bar{I}_h = \bar{Y}_{h1}\bar{V}_1 + \ldots + \bar{Y}_{hh}\bar{V}_h + \ldots + \bar{Y}_{hs}\bar{V}_s \\ \ldots \\ \bar{I}_s = \bar{Y}_{s1}\bar{V}_1 + \ldots + \bar{Y}_{sh}\bar{V}_h + \ldots + \bar{Y}_{ss}\bar{V}_s \end{cases}$$

in matrix formulation

$$\begin{bmatrix} \bar{I}_1 \\ \ldots \\ \bar{I}_h \\ \ldots \\ \bar{I}_s \end{bmatrix} = \begin{bmatrix} \bar{Y}_{11} & \ldots & \bar{Y}_{1h} & \ldots & \bar{Y}_{1s} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{Y}_{h1} & \ldots & \bar{Y}_{hh} & \ldots & \bar{Y}_{hs} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{Y}_{s1} & \ldots & \bar{Y}_{sh} & \ldots & \bar{Y}_{ss} \end{bmatrix} \cdot \begin{bmatrix} \bar{V}_1 \\ \ldots \\ \bar{V}_h \\ \ldots \\ \bar{V}_s \end{bmatrix} \Rightarrow [\mathbf{I}] = [\mathbf{Y}][\mathbf{V}]$$

$\mathbf{Y}$ is the so-called network **admittance matrix**

From the Physical Network to the Admittance Matrix – The Nodal Analysis

Properties of the nodal admittance matrix elements.

- a generic element $\bar{Y}_{i\ell}$ out of the main diagonal, called trans-admittance, is equal to the opposite of the admittance $\bar{y}_{i\ell}$ of the branch that connects the nodes i and ℓ:

$$\bar{Y}_{i\ell} = -\mathbf{y}_{i\ell} = \mathbf{I}_i \Big|_{\substack{\mathbf{V}_\ell = 1 \\ \mathbf{V}_h = 0 \quad \forall h \neq \ell}}$$

- a generic element $\bar{Y}_{ii}$ of the main diagonal, called self-admittance, is equal to the sum of all the admittances of the branches that are connected to node i including the ones with the neutral:

$$\bar{Y}_{ii} = \bar{y}_{io} + \sum \bar{y}_{i\ell} = \bar{I}_i \Big|_{\substack{\bar{V}_i = 1 \\ \bar{V}_\ell = 0 \quad \forall \ell \neq i}}$$

where the summation is extended to all of the nodes connected to node i.

From the Physical Network to the Admittance Matrix – The Nodal Analysis

Matrix $\mathbf{Y}$ is sparse. In case it is formed by **real values**, it is **diagonal-dominant** because each of its diagonal elements, in absolute value, is not lower than the sum of the other elements in the same row. In AC circuits, however, this property does not hold.

Matrix $\mathbf{Y}$ is also **symmetric** if all the two-port elements that compose the network are **reciprocal**.

Automated Construction of $\bar{\mathbf{Y}}$: Preliminaries

Consider a single-phase electrical network. Let the nodes be numbered as $n \in \mathcal{N} = \{1, \dots, N\}$, and the **ground** as $g \in \mathcal{G} = \{0\}$. Its topology is described by the **branches** $\ell_{\mathrm{k}} \in \mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ and the **shunts** $t_{\mathrm{n}} \in \mathcal{T} = \mathcal{N} \times \mathcal{G}$.

The branches are associated with the **longitudinal** electrical parameters (i.e., the **branch admittances** $\bar{Y}_{\ell_{\mathrm{k}}}$), and the shunts with the **transversal** electrical parameters (i.e., the **shunt admittances** $\bar{Y}_{t_{\mathrm{n}}}$) of the electrical network.

As we have seen, if the electrical network can be represented by a **lumped element model** (e.g., π-section equivalent circuits), then

- $\bar{Y}_{\ell_{\mathrm{k}}}$ of $\ell_{\mathrm{k}} = (m, n)$ is the sum of the admittances of the lumped elements connecting m and n (**branch elements**).
- $\bar{Y}_{t_{\mathrm{n}}}$ of $t_{\mathrm{n}} = (n, g)$ is the sum of the admittances of the lumped elements connecting n and g (**shunt elements**).

Automated Construction of $\bar{\mathbf{Y}}$: The Primitive Admittance Matrices $\bar{\mathbf{Y}}_{\mathcal{L}}$ and $\bar{\mathbf{Y}}_{\mathcal{T}}$

The branch admittances $\bar{Y}_{\ell_\mathrm{k}}$ ($\ell_\mathrm{k} \in \mathcal{L}$) compose the **primitive branch admittance matrix $\bar{\mathbf{Y}}_{\mathcal{L}}$**. This matrix is diagonal

$$\bar{\mathbf{Y}}_{\mathcal{L}} = \mathrm{diag}_{\ell_\mathrm{k} \in \mathcal{L}}\left(\bar{Y}_{\ell_\mathrm{k}}\right) = \mathrm{diag}\left(\left[\bar{Y}_{\ell_1}, \ldots, \bar{Y}_{\ell_{|\mathcal{L}|}}\right]\right) = \begin{bmatrix} \bar{Y}_{\ell_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{Y}_{\ell_{|\mathcal{L}|}} \end{bmatrix}$$

Analogously, the shunt admittances $\bar{Y}_{t_\mathrm{n}}$ ($t_\mathrm{n} \in \mathcal{T}$) compose the **primitive shunt admittance matrix $\bar{\mathbf{Y}}_{\mathcal{T}}$**, which is also diagonal

$$\bar{\mathbf{Y}}_{\mathcal{T}} = \mathrm{diag}_{t_\mathrm{n} \in \mathcal{T}}\left(\bar{Y}_{t_\mathrm{n}}\right) = \mathrm{diag}\left(\left[\bar{Y}_{t_1}, \ldots, \bar{Y}_{t_{|\mathcal{N}|}}\right]\right) = \begin{bmatrix} \bar{Y}_{t_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{Y}_{t_{|\mathcal{N}|}} \end{bmatrix}$$

Automated Construction of $\bar{\mathbf{Y}}$: The Incidence Matrix $\mathbf{A}_{\mathfrak{B}}$

The topology of the electrical grid is described by a directed graph $\mathfrak{B} = (\mathcal{N}, \mathcal{L})$, whose **vertices** are the nodes $\mathcal{N}$, and whose **edges** are the branches $\mathcal{L}$.

The connectivity of this graph is defined by the **edge-to-vertex incidence matrix** $\boldsymbol{A}_{\mathfrak{B}}$, whose elements a_{kn} are given as follows

$$a_{kn} = \begin{cases} +1 \ (\text{if } \ell_{\text{k}} = (n,\cdot) \in \mathcal{L}) \\ -1 \ (\text{if } \ell_{\text{k}} = (\cdot, n) \in \mathcal{L}) \\ \ 0 \qquad (\text{otherwise}) \end{cases}$$

That is, $a_{kn} = +1$ if the branch ℓ_{k} **originates at the node** n, and $a_{kn} = -1$ if the branch ℓ_{k} **terminates at the node** n. Observe that the rows of $\mathbf{A}_{\mathfrak{B}}$ correspond to the vertices/branches, and the columns to the vertices/nodes.

Automated Construction of $\bar{\mathbf{Y}}$: The Incidence Matrix $\mathbf{A}_{\mathfrak{B}}$

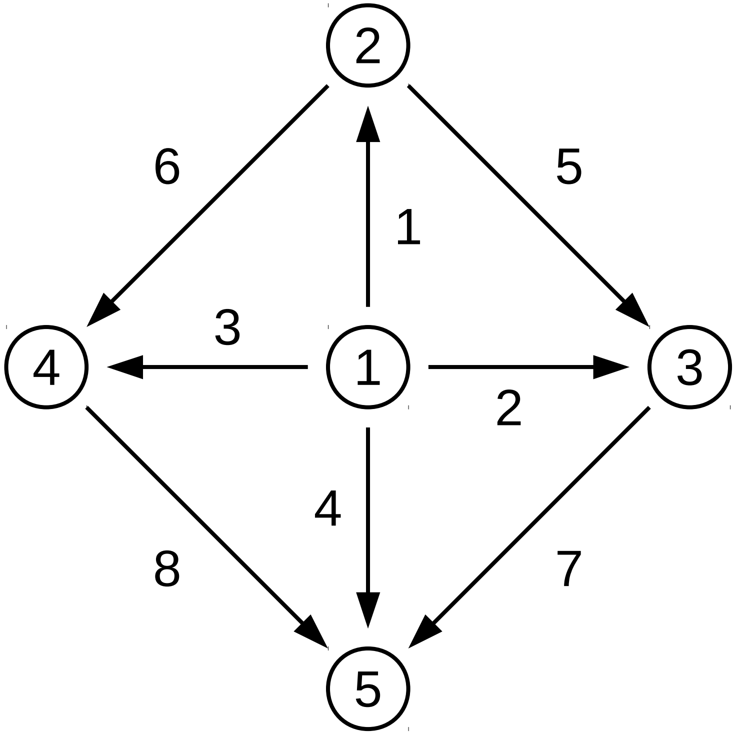
Consider the example graph shown on the right-hand side.

It has the following incidence matrix

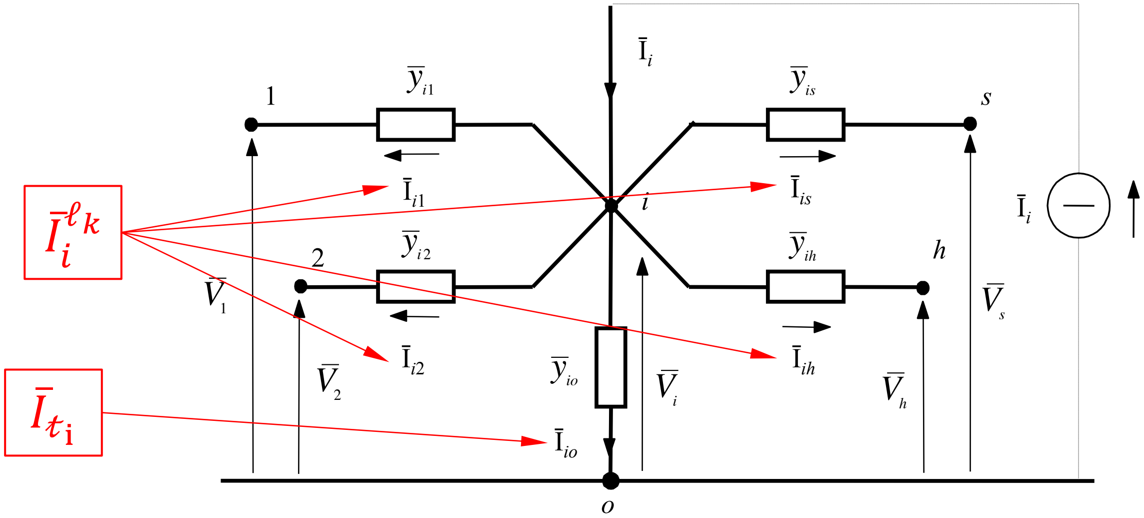
$$\mathbf{A}_{\mathfrak{B}} = \begin{array}{c} \textbf{Nodes} \\ \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & \\ \end{array} \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \text{Branches} \\ +1 & -1 & 0 & 0 & 0 & 1 \\ +1 & 0 & -1 & 0 & 0 & 2 \\ +1 & 0 & 0 & -1 & 0 & 3 \\ +1 & 0 & 0 & 0 & -1 & 4 \\ 0 & +1 & -1 & 0 & 0 & 5 \\ 0 & +1 & 0 & -1 & 0 & 6 \\ 0 & 0 & +1 & 0 & -1 & 7 \\ 0 & 0 & 0 & +1 & -1 & 8 \end{array}$$

$$\mathbf{A}_{\mathfrak{B}} = \begin{bmatrix} +1 & -1 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 \end{bmatrix}$$



Automated Construction of $\bar{\mathbf{Y}}$: The Nodal Admittance Matrix $\mathbf{Y}$



We can rewrite the Kirchhoff's current law applied to the generic node i of the grid as follows:

$$\bar{I}_i = \sum_k \bar{I}_{i,out}^{\ell_k} - \sum_k \bar{I}_{i,in}^{\ell_k} + \bar{I}_{t_\mathrm{i}}$$

where $\bar{I}_{i,out}^{\ell_k}$ is the "inner current" of Π-equivalent branch ℓ_k leaving node i and $\bar{I}_{i,in}^{\ell_k}$ the "inner current" of Π-equivalent branch ℓ_k entering node i and $\bar{I}_{t_\mathrm{i}} = \bar{I}_{io}$ (rename of current flowing through the sum of shunts on node i.

Automated Construction of $\bar{\mathbf{Y}}$: The Nodal Admittance Matrix $\mathbf{Y}$

Therefore, from the definition of the incidence matrix $\mathbf{A}_{\mathfrak{B}}$, we have:

$$\bar{\mathbf{I}} = \mathbf{A}_{\mathfrak{B}}^{T}\bar{\mathbf{I}}_{\mathcal{L}} + \bar{\mathbf{I}}_{\mathcal{T}}$$

Where $\bar{\mathbf{I}}_{\mathcal{L}}$ is the array of branch currents $\bar{I}_{i}^{\ell_k}$ and $\bar{\mathbf{I}}_{\mathcal{T}}$ the array of shunt currents $\bar{I}_{t_{\mathrm{i}}}$.

Now, from Kirchhoff's voltage law applied to ℓ_k we can write:

$$\bar{I}_{i}^{\ell_k} = \bar{Y}_{\ell_k}(\bar{V}_i - \bar{V}_h)$$

and for all the branches we have:

$$\bar{\mathbf{I}}_{\mathcal{L}} = \bar{\mathbf{Y}}_{\mathcal{L}}\mathbf{A}_{\mathfrak{B}}\bar{\mathbf{V}}$$

For the shunts we can write $\bar{\mathbf{I}}_{\mathcal{T}}$ as follows:

$$\bar{\mathbf{I}}_{\mathcal{T}} = \bar{\mathbf{Y}}_{\mathcal{T}}\bar{\mathbf{V}}$$

Therefore, we have:

$$\bar{\mathbf{I}} = \mathbf{A}_{\mathfrak{B}}^{T}\bar{\mathbf{I}}_{\mathcal{L}} + \bar{\mathbf{I}}_{\mathcal{T}} = \left(\mathbf{A}_{\mathfrak{B}}^{T}\bar{\mathbf{Y}}_{\mathcal{L}}\mathbf{A}_{\mathfrak{B}} + \bar{\mathbf{Y}}_{\mathcal{T}}\right)\bar{\mathbf{V}} \rightarrow \bar{\mathbf{Y}} = \mathbf{A}_{\mathfrak{B}}^{T}\bar{\mathbf{Y}}_{\mathcal{L}}\mathbf{A}_{\mathfrak{B}} + \bar{\mathbf{Y}}_{\mathcal{T}}$$

Automated Construction of $\overline{\mathbf{Y}}$: The Nodal Admittance Matrix $\mathbf{Y}$

$$\overline{\mathbf{Y}} = \mathbf{A}_{\mathfrak{B}}^{T}\overline{\mathbf{Y}}_{\mathcal{L}}\mathbf{A}_{\mathfrak{B}} + \overline{\mathbf{Y}}_{\mathcal{T}}$$

Note that $\mathbf{A}_{\mathfrak{B}}$ describes the topology of the electrical network, while $\overline{\mathbf{Y}}_{\mathcal{L}}$ and $\overline{\mathbf{Y}}_{\mathcal{T}}$ describe its electrical properties. A change in the topology only affects $\mathbf{A}_{\mathfrak{B}}$, and a change in the parameters only affects $\overline{\mathbf{Y}}_{\mathcal{L}}$ and/or $\overline{\mathbf{Y}}_{\mathcal{T}}$. If any of these matrices changes during the operation, $\overline{\mathbf{Y}}$ can be updated easily using the above-stated formula.

Automated Construction of $\bar{\mathbf{Y}}$: The Nodal Admittance Matrix $\bar{\mathbf{Y}}$

The **diagonal elements** $\bar{Y}_{nn}$ $(n \in \mathcal{N})$ of $\bar{\mathbf{Y}}$ are given by

$$\bar{Y}_{nn} = \bar{Y}_{t_\mathrm{n}} + \sum_{\ell_\mathrm{k}=(n,\cdot)} \bar{Y}_{\ell_\mathrm{k}} + \sum_{\ell_\mathrm{k}=(\cdot,n)} \bar{Y}_{\ell_\mathrm{k}}$$

That is, the diagonal element at position n is the sum of all branch/shunt admittances connected to the respective node.

The **off-diagonal elements** $\bar{Y}_{mn}$ $(m, n \in \mathcal{N}, m \neq n)$ are given by

$$\bar{Y}_{mn} = \begin{cases} -\bar{Y}_{\ell_\mathrm{k}} & (\exists \ell_\mathrm{k} = (m,n) \text{ or } \exists \ell_\mathrm{k} = (n,m)) \\ 0 & (\text{otherwise}) \end{cases}$$

That is, the off-diagonal element at position m, n is the negative of the corresponding branch admittance (if the branch exists).

Analysis of Asymmetrical Power Systems: Introduction

As discussed before, **balanced power systems**, whose lines are balanced and transposed, and whose loads are symmetrical, can be represented by equivalent **positive-sequence** circuits.

However, **power distribution systems** may be characterized by **unbalanced non-transposed lines** and **asymmetrical loads**. Therefore, they cannot be represented by positive-sequence components only.

In an asymmetrical power system, the sequence networks are coupled. That is, the flow of current in one sequence gives rise to voltage drops in all sequences. Therefore, the **method of symmetrical components** cannot be used.

Analysis of Asymmetrical Power Systems: Phase-Domain Model

Consider an **unbalanced three-phase network.** As the method of symmetrical components cannot be used, the three phases (denoted by a, b, c) need to be treated explicitly.

The electrical network is still described by an equation of the form $\bar{\mathbf{I}} = \bar{\mathbf{Y}}\bar{\mathbf{V}}$. The vectors $\bar{\mathbf{I}}$ and $\bar{\mathbf{V}}$ contain the nodal current and voltage phasors of all phases in all nodes. Namely

$$\bar{\mathbf{I}} = \begin{bmatrix} \bar{\mathbf{I}}_1 \\ \vdots \\ \bar{\mathbf{I}}_N \end{bmatrix} = \begin{bmatrix} \bar{I}_{1,a} \\ \bar{I}_{1,b} \\ \bar{I}_{1,c} \\ \vdots \\ \bar{I}_{N,a} \\ \bar{I}_{N,b} \\ \bar{I}_{N,c} \end{bmatrix}, \bar{\mathbf{V}} = \begin{bmatrix} \bar{\mathbf{V}}_1 \\ \cdots \\ \bar{\mathbf{V}}_N \end{bmatrix} = \begin{bmatrix} \bar{V}_{1,a} \\ \bar{V}_{1,b} \\ \bar{V}_{1,c} \\ \vdots \\ \bar{V}_{N,a} \\ \bar{V}_{N,b} \\ \bar{V}_{N,c} \end{bmatrix}$$

In this case, $\bar{\mathbf{Y}}$ is called **compound nodal admittance matrix**.

Analysis of Asymmetrical Power Systems: Compound Electrical Parameters

The branches and shunts of a three-phase network can be represented using **compound electrical parameters.**

Namely, **branch admittance matrices** $\bar{\mathbf{Y}}_{\ell_{\mathrm{k}}}$ $(\ell_{\mathrm{k}} \in \mathcal{L})$, where

$$\bar{\mathbf{Y}}_{\ell_{\mathrm{k}}} = \begin{bmatrix} \bar{Y}_{\ell_{\mathrm{k}},aa} & \bar{Y}_{\ell_{\mathrm{k}},ab} & \bar{Y}_{\ell_{\mathrm{k}},ac} \\ \bar{Y}_{\ell_{\mathrm{k}},ba} & \bar{Y}_{\ell_{\mathrm{k}},bb} & \bar{Y}_{\ell_{\mathrm{k}},bc} \\ \bar{Y}_{\ell_{\mathrm{k}},ca} & \bar{Y}_{\ell_{\mathrm{k}},cb} & \bar{Y}_{\ell_{\mathrm{k}},cc} \end{bmatrix}$$

and **shunt admittance matrices** $\bar{\mathbf{Y}}_{\mathscr{t}_{\mathrm{n}}}$ $(\mathscr{t}_{\mathrm{n}} \in \mathcal{T})$, where

$$\bar{\mathbf{Y}}_{\mathscr{t}_{\mathrm{n}}} = \begin{bmatrix} \bar{Y}_{\mathscr{t}_{\mathrm{n}},aa} & \bar{Y}_{\mathscr{t}_{\mathrm{n}},ab} & \bar{Y}_{\mathscr{t}_{\mathrm{n}},ac} \\ \bar{Y}_{\mathscr{t}_{\mathrm{n}},ba} & \bar{Y}_{\mathscr{t}_{\mathrm{n}},bb} & \bar{Y}_{\mathscr{t}_{\mathrm{n}},bc} \\ \bar{Y}_{\mathscr{t}_{\mathrm{n}},ca} & Y_{\mathscr{t}_{\mathrm{n}},cb} & \bar{Y}_{\mathscr{t}_{\mathrm{n}},cc} \end{bmatrix}$$

These matrices are usually **symmetric** or **circulant.**

Analysis of Asymmetrical Power Systems: Compound Nodal Admittance Matrix

In analogy to the single-phase case, define the **block matrices**

$$\bar{\mathbf{Y}}_{\mathcal{L}} = \mathrm{diag}_{\ell_\mathrm{k}\in\mathcal{L}}\left(\bar{\mathbf{Y}}_{\ell_\mathrm{k}}\right) = \mathrm{diag}\left(\left[\bar{\mathbf{Y}}_{\ell_1}, \dots, \bar{\mathbf{Y}}_{\ell_{|\mathcal{L}|}}\right]\right)$$

$$\bar{\mathbf{Y}}_{\mathcal{T}} = \mathrm{diag}_{t_\mathrm{n}\in\mathcal{T}}\left(\bar{\mathbf{Y}}_{t_\mathrm{n}}\right) = \mathrm{diag}\left(\left[\bar{\mathbf{Y}}_{t_1}, \dots, \bar{\mathbf{Y}}_{t_\mathrm{N}}\right]\right)$$

$$\mathbf{A}_{\mathfrak{B},\mathrm{kn}} = \begin{cases} +\mathbf{I}_3 & (\text{if } \ell_\mathrm{k} = (n,\cdot) \in \mathcal{L}) \\ -\mathbf{I}_3 & (\text{if } \ell_\mathrm{k} = (\cdot, n) \in \mathcal{L}) \\ 0 & (\text{otherwise}) \end{cases}$$

Where $\mathbf{I}_3$ is an identity matrix of size 3×3. Then, $\bar{\mathbf{Y}}$ can still be computed using the formula

$$\bar{\mathbf{Y}} = \mathbf{A}_{\mathfrak{B}}^{T}\bar{\mathbf{Y}}_{\mathcal{L}}\mathbf{A}_{\mathfrak{B}} + \bar{\mathbf{Y}}_{\mathcal{T}}$$

The only difference to the single-phase case is that $\bar{\mathbf{Y}}_{\mathcal{L}}$, $\mathbf{A}_{\mathfrak{B}}$, and $\bar{\mathbf{Y}}_{\mathcal{T}}$ are now **block matrices**.